Deforming motivic theories I: Pure weight perfect Modules on divisorial schemes

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Abstract

In this paper, we introduce a notion of weight r pseudo-coherent Modules associated to a regular closed immersion $i : Y \hookrightarrow X$ of codimension r, and prove that there is a canonical derived Morita equivalence between the DG-category of perfect complexes on a divisorial scheme X whose cohomological support are in Y and the DG-category of bounded complexes of weight r pseudo-coherent $\mathcal{O}_X$-Modules supported on Y. The theorem implies that there is the canonical isomorphism between the Bass-Thomason-Trobaugh non-connected K-theory [TT90], [Sch06] (resp. the Keller-Weibel cyclic homology [Kel98], [Wei96]) for the immersion and the Schlichting non-connected K-theory [Sch04] associated to (resp. that of) the exact category of weight r pseudo-coherent Modules. For the connected K-theory case, this result is just Exercise 5.7 in [TT90]. As its application, we will decide on a generator of the topological filtration on the non-connected K-theory (resp. cyclic homology theory) for affine Cohen-Macaulay schemes.

1 Introduction

Since the word "motive theory" is an ambiguous word, in this Introduction, as motive theory, we restrictedly mean axiomatic studying (co)homology theories over algebraic varieties by enriching morphisms between algebraic varieties with adequate equivalence relations. Traditionally, to construct motivic

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categories, we used to choose certain classes of algebraic cycles as morphisms spaces and consider various equivalence relations on them, for example rational, numerical and algebraic relations and so on. In practice, the difficulty of handling a motivic theory is concentrating on moving algebraic cycles suitably in an appropriate equivalence relation class (see the proficient survey [Lev06]). A problem of this type is so-called "moving lemma" and solving by deliberating on geometry over a base (field). In this paper, we give a first step of building up a motivic theory which does not rely upon geometry over a base by replacing (moduli spaces of) algebraic cycles with (roughly speaking, moduli non-commutative spaces of) pseudo-coherent complexes and considering an equivalence relation on them as the derived Morita equivalences (Compare [Kon07] §4, [Tau07]).

The aim of this paper is to introduce the notion of (*Thomason-Trobaugh*) *weight* on the class of perfect Modules on schemes inspired by the work of Thomason and Trobaugh in [TT90]. To explain this more precisely, let X be a divisorial scheme (in the sense of [BGI71], cf. Def. 3.12) and $i : Y \hookrightarrow X$ a regular closed immersion of codimension r. A pseudo-coherent $\mathcal{O}_X$-Module is said to be *of* (*Thomason-Trobaugh*) *weight* r supported on Y if it is of Tor-dimension $\leqq r$ and supported on Y. Here the word "weight" is coming from the weight of the Adams operations in [GS87] and a more systematic study will be done in [Moc08]. We denote by $\mathbf{Wt}^r(X \text{ on } Y)$ the exact category of pseudo-coherent $\mathcal{O}_X$-Modules of weight r supported on the subspace Y and $\mathbf{Perf}(X \text{ on } Y)$ the exact category of perfect complexes on X whose cohomological support are in Y. We shall prove the following theorem:

Theorem (Th. 4.3)**.** *There is a canonical derived Morita equivalence between the exact category of bounded complexes of* $\mathbf{Wt}^r(X \text{ on } Y)$ *and* $\mathbf{Perf}(X \text{ on } Y)$.

As alluded to above, it can be considered as one of a variant of "moving lemma". It might sound a new flavored theory, but the methods of proving Theorem 4.3 are classical, standard and almost all of them were established by Grothendieck school. For example, Verdier's coherator theory (Prop. 3.6), Ilusie's global resolution theorem (Th. 3.15), Grothendieck's local cohomology theory (Lem. 5.9) and so on. The theorem implies that there is a canonical isomorphism between the Bass-Thomason-Trobaugh non-connected K-theory $K^B(X \text{ on } Y)$ [TT90], [Sch06] (resp. the Keller-Weibel cyclic homology $HC(X \text{ on } Y)$ [Kel98], [Wei96]) and the Schlichting non-connected K-theory [Sch04] associated to (resp. that of) the exact category of weight r

pseudo-coherent $\mathcal{O}_X$-Modules $K^S(\mathbf{Wt}(X \text{ on } Y))$ (resp. $HC(\mathbf{Wt}(X \text{ on } Y))$). That is, we have isomorphisms

$$K^B_q(X \text{ on } Y) \simeq K^S_q(\mathbf{Wt}^r(X \text{ on } Y)),$$
$$HC_q(X \text{ on } Y) \simeq HC_q(\mathbf{Wt}^r(X \text{ on } Y)),$$

for each $q \in \mathbb{Z}$. For the connected K-theory this result is nothing other than Exercise 5.7 in [TT90]. For Grothendieck groups ($q = 0$), there is a detailed proof if X is the spectrum of a Cohen-Macaulay local ring and Y is the closed point of X ([RS03], Prop. 2). For K-theory, as mentioned in Exercise 5.7, this problem is related with the works [Ger74], [Gra76] and [Lev88]. Namely the problem about describing the homotopy fiber of $K^B(X) \to K^B(X \smallsetminus Y)$ (or rather than $K^Q(X) \to K^Q(X \smallsetminus Y)$) by using the K-theory of a certain exact category. As described in [Ger74], there is an example due to Deligne which suggests difficulty of the problem for a general closed immersion. Conversely, the example indicate that for an appropriate scheme X, there is a good class of pseudo-coherent $\mathcal{O}_X$-Modules. That is, Modules of pure weight. This concept is intimately related to Weibel's K-dimensional conjecture [Wei80] (see Conj. 6.4), Gersten's conjecture [Ger73] and its consequences. These subjects will be treated in [HM08], [Moc08]. Notice that there are different notions of pure weight by Grayson [Gra95] and Walker [Wal00] and these two notions are compatible in a particular situation [Wal96]. In a future work, the authors hope to compare the Walker weight with the Thomason-Trobaugh one by utilizing the (*equidimensional*) *bivariant algebraic K-theory* [GW00].

Now we explain the structure of the paper. In §2, we describe to our motivational picture. After reviewing the fundamental facts in §3, we will define the notion of weight and state the main theorem in §4. The proof of the main theorem will be given in §5. Finally we will give applications of the main theorem in §6.

Convention. Throughout this paper, we use the letter X to denote a scheme. A *complex* means a chain complex whose boundary morphism is increase level of term by one. For fundamental notations of chain complexes, for example mapping cone and mapping cylinder etc..., we follow the book [Wei94]. For an additive category $\mathcal{A}$, we denote by $\mathbf{Ch}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. The word "$\mathcal{O}_X$-Module" means a sheaf on X which is a sheaf of modules over the sheaf of rings $\mathcal{O}_X$. We denote by $\mathbf{Mod}(X)$ the abelian category of $\mathcal{O}_X$-Modules and $\mathbf{Qcoh}(X)$ the category of quasi-coherent $\mathcal{O}_X$-Modules. An *algebraic vector bundle* over the scheme X is a locally free $\mathcal{O}_X$-Module of

finite rank and we denote by $\mathcal{P}(X)$ the category of algebraic vector bundles. In particular a *line bundle* is an algebraic vector bundle of rank one (= an invertible sheaf). For the terminologies of algebraic K-theory, we follow to the notations in [Sch07]. For example, for a complicial biWaldhausen category $\mathcal{C}$, we denote its associated derived category by $\mathcal{T}(\mathcal{C})$ and for an exact category $\mathcal{E}$, we denote its associated derived category $\mathcal{T}(\mathbf{Ch}(\mathcal{E}))$ by $\mathcal{D}(\mathcal{E})$. Finally for the $\mathbb{A}^1$-motivic theory, we follow the notations in [MVW06].

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2 Conjectural picture

In this section, we will give a conjectural perspective of *deforming motivic theories*. This section is logically independent of the others.

2.1 Analogies between multiplicative and additive motivic theories

As in the Introduction, as a motive theory, we prefer to mean axiomatic studying of (co)homology theories over algebraically geometric objects by enriching morphisms between algebraically geometric objects with adequate equivalence relations. So there should be many motivic theories depending on our treating of algebraically geometric objects and (co)homology theories. For example, if we deal with Weil cohomology theories, the classical motive theory is fitting for our purpose [Kle68]. If we handle $\mathbb{A}^1$-homotopy invariant (co)homology theories, the motivic homotopy theory in the sense of Voevodsky is appropriate [Voe00]. If we consider cohomology theories which has the Gersten resolution, the Bloch-Ogus(-Gabber) theory [BO74], [CHK97] is suitable. Moreover there are other motivic theories for example [KS02], [KL07]. It might be believed that there is "the" motive theory which is omniscient and unifying every motivic theories. But as in the following example, there are motivic theories which are not seemed to be compatible with each other.

Example 2.1. If we prefer to give a motivic interpretation of the Hodge decomposition using the cyclic homology theory like as [Wei97], or if we like to understand what is the motive associated with the additive group $\mathbf{G}_a$ like as a generalized 1-motive [Lau96], [Ber08], we shall not realize them in Voevodsky's motivic world. For the cyclic homology theory is not $\mathbb{A}^1$-homotopy invariant and $\mathbf{G}_a$ is contractible in his motivic category. But we have the analogies table between additive and multiplicative worlds as in [Lod03]. We would like to extend the table to motivic stage. For example, the Bloch theorem [Blo86] and the Hodge decomposition as in the table below.

$\times$ (Voevodsky's motivic theory)	$+$ (additive motivic theory)
$K_n(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{p+q=n} \mathrm{H}^p_{\mathcal{M}}(X, \mathbf{G}_m^{\otimes q})_{\mathbb{Q}}$	$\mathrm{H}^n(X, \mathbb{C}) \xrightarrow{\sim} \bigoplus_{p+q=n} \mathrm{H}^p(X, \mathbf{G}_a \otimes \mathbf{G}_m^{\otimes q})$

Of course, the right hand side above is conjectural description. (But see [BE03], [Rul07], [Par07] and [Par08]). In these analogical line, following [FT85] and [FT87], we like to call the cyclic homology theory the *additive algebraic K-theory*.

We shall also notice the fact that there are real mathematical problems stretching away both additive and multiplicative worlds. For example, Vorst's conjecture [Vor79]. Actually the conjecture is proved in a special case by frequently utilizing both multiplicative and additive motivic techniques [CHW06]. In the next subsection we propose another similarly kind problems.

2.2 Motivic modules and Weil reciprocity law

Classically there is the following problem.

Problem 2.2. Let $G_1, \ldots, G_r$ be commutative group varieties over a base field k. Then we have the correspondence

$$Z_0(G_1 \times_k G_2 \times_k \cdots \times_k G_r) \leftrightarrow \bigoplus_{L/k:\substack{\text{finite}\\ \text{extension}}} G_1(L) \otimes_{\mathbb{Z}} G_2(L) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} G_r(L)$$

where $Z_0(?)$ means the group of zero cycles. The problem is the following: *What are the suitable equivalence relations making assignment above isomorphism.*

$$Z_0(G_1 \times_k G_2 \times_k \cdots \times_k G_r)/\sim \xrightarrow{\sim} \bigoplus_{L/k:\substack{\text{finite}\\ \text{extension}}} G_1(L) \otimes_{\mathbb{Z}} G_2(L) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} G_r(L)/\sim .$$

Historical Note 2.3. If we assume all $G_1, \ldots, G_r$ are semi-abelian varieties, then there are suitable candidates for equivalence relations above.
(i) In the left hand side, the suitable equivalence relation should come from the tensor products as 1-motives in the sense of [Del74]. That is, the left hand side should be replaced with

$$\Gamma(\mathrm{Spec}\, k, G_1 \otimes \cdots \otimes G_r) = Z_0(G_1 \times_k G_2 \times_k \cdots \times_k G_r)/\sim$$

where tensor product are taken as 1-motives.
(ii) In the right hand side, Kazuya Kato proposed that the suitable equivalence relation should be the following two relations.
• Projection formula for norms.
• Weil reciprocity law for semi-abelian varieties.

We will write the left hand side modulo equivalence relations above as

$$K(k, G_1, \ldots, G_r)$$

and called it *Milnor K-group associated with $G_1, \ldots, G_r$* (see for example [Som90], [Kah92]). The naming coming from the following isomorphism.

$$K(k, \overbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}^{r}) \xrightarrow{\sim} K_r^M(k).$$

Observations 2.4. (i) (At least after tensoring with $\mathbb{Q}$,) the tensor product as 1-motives is equal to the tensor product in the $\mathbb{A}^1$-motivic category $\mathrm{DM}(k)$ (see for example [Org04], [BK07]).
(ii) The projection formula relation above is one of the consequence of presheaf with transfer, that is, there is the following statement (see for example [Org04]):
Every commutative group variety over a field k is considered as a functor

$$\mathbf{qpsmcor}(k) \to \mathbf{Ab}$$

where $\mathbf{qpsmcor}(k)$ *is the category of quasi-projective smooth varieties whose morphisms are finite surjective correspondences.*
(iii) If all G_i are 1-dimensional semi-abelian varieties, we have the following formula,

$$\Gamma(\mathrm{Spec}\, k, G_1 \otimes, \cdots \otimes G_r) \xrightarrow{\sim} K(k, G_1, \ldots, G_r).$$

This is the affirmative answer for question above (see for $\mathbf{G}_m$ case [SV00] and for elliptic curve case [Moc06]).

(iv) The reason why semi-abelian varieties are fit in Voevodsky's theory is that semi-abelian varieties are $\mathbb{A}^1$-homotopy invariant presheaves with transfers. So we shall say $\mathbb{A}^1$-homotopy invariant presheaves with transfers as a *motivic modules*. Then we can re-write the statement in Historical Notes 2.3 (i) as follows.

In the left hand side, the suitable equivalence relation should come from the tensor products as motivic modules.

In the observation and § 2.1, we are interested in the following question. *What is a good notion of motivic modules including* $\mathbf{G}_a$*?*

Remark 2.5. (i) (cf. [RO06]) The category of motives is reinterpreted in the context of stable motivic homotopy theory by Röndings and Østvær as follows. Let $\mathbb{MZ} \in \mathbf{SH}^{\mathbb{A}^1}(k)$ be the motivic Eilenberg-Maclane spectrum. Then $\mathbb{MZ}$ is considered as a ring object in $\mathbf{SH}^{\mathbb{A}^1}(k)$ in the natural way and we have the following identity:

$$\mathbf{Mod}(\mathbb{MZ}) \xrightarrow{\sim} \mathrm{DM}(k)$$

where we assume that characteristic of k is zero. This means $\mathrm{DM}(k)$ is actually "the category of motivic modules" in some sense. Notice that if a presheaf of abelian groups on the category of quasi-projective smooth schemes has an action of $\mathbb{MZ}$, this means that F can extend to a presheaf on $\mathbf{qpsmcor}(k)$.

(ii) Several authors are attempting to describe $\Gamma(\operatorname{Spec} k, G_1 \otimes \cdots \otimes G_r)$ as generators and relations. In this point, relations are related with the functional equations of special functions associated with G_i. For example, if all G_i are equal to $\mathbf{G}_m$, the special function is the polylogarithms [Gon94] and so on. Therefore it is quite surprised that the relations of $K(k, G_1, \ldots, G_r)$ does not depend on the G_i. The Weil reciprocity law is implicitly controlling the functional equations of special functions associated with G_i. So it is important that we shall ask what is a meaning of the Weil reciprocity law in the context of Voevodsky's motivic theory.

We can state a generalization of the Weil reciprocity law which is called *Motivic reciprocity law*. Let k be a field which satisfies the resolution of singularity assumption.

Theorem 2.6 ([Moc06]). *For a field extension of transcendental degree one K/k, the composition of*

$$\text{(1)}\quad \mathrm{M}(\operatorname{Spec} k)(1)[1] \overset{\Sigma N_{k(v)/k}(1)[1]}{\to} \tilde{\prod_{\substack{v:\text{place}\\ \text{of } K/k}}} \mathrm{M}(\operatorname{Spec} k(v))(1)[1] \overset{\tilde{\prod}\partial_v}{\to} \mathrm{M}(\operatorname{Spec} K)$$

is the zero map in the pro-category of $DM(k)$.

If we take the $\mathrm{Hom}_{\mathrm{DM}(k)}(?, \mathbb{Z}(n+1)[n+1])$ for the sequence (1), we can easily reprove the Weil reciprocity law for Milnor K-groups.

Corollary 2.7 ([Sus82]). *The composition of*

$$K^M_{n+1}(K) \overset{\oplus\partial_v}{\to} \bigoplus_{\substack{v:\text{place}\\ \text{of } K/k}} K^M_n(k(v)) \overset{\Sigma N_{k(v)/k}}{\to} K^M_n(k)$$

is the zero map.

The crucial point of proving the motivic reciprocity law is the existence of functorial Gysin triangles which is proved by Déglise [Deg06] and $\mathbb{A}^1$-homotopy invariance is indispensable in his construction of the triangle. On the other hand, Rülling proved the Weil reciprocity law for the de Rham-Witt complexes which is not an $\mathbb{A}^1$-homotopy invariant theory [Rul07]. We would like to explain this reciprocity law also in the context of an alien motivic theory. In this way, we sometimes have interested in the problems which stretching away several motivic theories and sometimes intend to analyze relationship of several motivic theories, for example, their analogies and differences. The main theme of *deforming motivic theories* is investigating the relationship between various motivic theories. In particular, Voevodsky's motivic theory and an alien (additive) motivic theory.

2.3 How to describe deforming motivic theories I

Next we intend to illustrate how to describe deforming motivic theories. As in [Han95], [RO06] and [BV07], the triangulated category of motivic sheaves shall be the connected components of the ∞-category of that in some sense. Here the word "∞-category" means (quasi-) DG-category or $\mathcal{S}$-category in the sense of Töen and Vezzosi [TV04]. We first start to consider how to mention an alien motivic theory as follows.

Example 2.8. (i) (Toy model) Let V be a finite dimensional vector space over a field k with an inner product and W its sub vector space. Then we have an isomorphism

$$V/W \xrightarrow{\sim} W^{\perp} \tag{2}$$

where $W^{\perp}$ is the orthogonal subspace of W in V.
(ii) Let k be a perfect field, V the derived category of complexes of Nisnevich sheaves transfer over k bounded from above and W the localizing subcategory generated by the complexes of the form

$$\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$$

for smooth schemes X over k. Then we have the equivalence (2) where $W^{\perp}$ is the full subcategory of those complexes whose cohomology sheaves are $\mathbb{A}^1$-homotopy invariant in V (see [Voe00], Prop. 3.2.3). The sign $\perp$ is justified in the context of (generalized) topoi theory or Bousfiled localization theory as below.
(iii) (cf. [BGV72], IV) Let $\mathcal{C}$ be a small category with a Grothendieck topology τ. We denote the category of presheaves on $\mathcal{C}$ by V and the category of τ-local contractible presheaves on $\mathcal{C}$ by W. Then we have an equivalence (2) where $W^{\perp}$ is the full subcategory of τ-sheaves in V. Namely, an object F in $W^{\perp}$ is satisfying the decent condition (or rather than saying the orthogonal condition) as follows:

$$\mathrm{Hom}(\mathfrak{U}, F) \xrightarrow{\sim} \mathrm{Hom}(h_X, F)$$

where h_X is the functor represented by an object X in $\mathcal{C}$ and $\mathfrak{U}$ is a crible in $\tau(X)$. As in [Hir03], [TV05], replacing $\mathcal{C}$ as above with a more higher categorical (or rather than say homotopical) object in some sense, the argument above still works fine by replacing the decent condition with the hyper one (For precise statement, consult with [TV05]). For DG-categories case, see [Dri04] and [Tau07] Appendix.

Now we would better consider the reason why a hyper descent condition is not seemed to be involved in Voevosky's $\mathbb{A}^1$-homotopy theory. To do so, let us recall the following Lemma 2.10:

Definition 2.9. Let (I, x, y) be a triple consisting of $I \in \mathbf{qpsmcor}(k)$ and different k-rational points $x : \operatorname{Spec} k \to I$, $y : \operatorname{Spec} k \to I$.

(i) Two maps $f,\, g : X \to Y$ in $\mathbf{qpsmcor}(k)$ are said to be *I-homotopic* if there is a map $H : X \times I \to Y$ such that $H \circ x \times \mathrm{id}_X = f$ and $H \circ y \times \mathrm{id}_Y = g$ (or $H \circ x \times \mathrm{id}_X = g$ and $H \circ y \times \mathrm{id}_Y = f$).

(ii) A functor $F : \mathbf{qpsmcor}(k) \to \mathcal{C}$ is said to be *I-homotopy invariant* if for any I-homotopic maps $f,\, g : X \to Y$, $F(f) = F(g)$.

For $\mathbb{A}^1$-homotopy invariant, we mean that $\mathbb{A}^1$-homotopy invariant for the triple $(\mathbb{A}^1, 0, 1)$.

Lemma 2.10. *For any presheaf F on $\mathbf{qpsmcor}(k)$, the following conditions are equivalent.*

(i) *For any scheme X, the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism*

$$F(X \times \mathbb{A}^1) \xrightarrow{\sim} F(X).$$

(ii) *F is $\mathbb{A}^1$-homotopy invariant.*

Notice that the condition (i) (resp. (ii)) above is a descent condition for objects (=0-morphisms) (resp. morphisms (=1-morphisms)) in some sense. Therefore the condition (i) is seemed to be stronger than the condition (ii). We designate that to prove (i) from the condition (ii), we are using the special feature of $\mathbb{A}^1$. Namely the existence of the multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \ni (x, y) \mapsto xy \in \mathbb{A}^1$ and this feature is axiomized by Voevodsky as the site with interval theory [Voe96], [MV99]. In the authors view point, this is the reason why we are able to shortcut to construct the motivic homotopy category without using a hyper descent theory and there is no reason that to establish an alien motivic category, we can avoid using a higher topoi theory. So we propose the following.

Conjecture 2.11 (Very obscure version)**.** *To build up an alien motivic category, we need to choose a moduli space V of algebraically geometric objects which could be represented by ∞-category or homotopical category as a generator class and a moduli space W of relations space. Then we can define an alien motivic category by the quotient space V/W and somewhat hyper descent theory implies that it is equivalent to $W^{\perp}$. Here $W^{\perp}$ is full subspace of V consisting of the objects which satisfy orthogonal condition in some sense.*

Obviously the conjecture has two faces. One face is the problem of establishing the general frame works of a higher or generalized topoi theory

fitting for our purpose. For example, presheaves with transfer theory and site with interval theory can be considered as a sheave theory over generalized Grothendieck topology and are suitable for describing $\mathbb{A}^1$-motivic theory. The other face is the problem of finding the good class of V and W above. To attack the first face, we need drastically axiomatic consideration. To study the second one, we need look squarely at real many examples. The authors are starting from attacking to the second one. After getting many important examples, they intend to contemplate the first one [HM08].

2.4 Thomason categories and bivariant algebraic K-theory

It is a complicial biWaldhausen category closed under the formation of the canonical homotopy push-outs and pull-backs in the sense of [TT90] that makes sense of its derived category and algebraic (resp. additive) K-theory and we like to call it a *Thomason category*. A morphism between Thomason categories is complicial exact functor in the sense of *op. cit.* In this paper, we examine V in Conjecture 2.11 as the category of Thomason categories which is a homotopical category in the sense of [DHKS04] by declaring the class of weak equivalences as *derived Morita equivalences*, that is, morphisms which induce equivalences of derived categories. The reasons why we prefer to take algebraically geometric objects as Thomason categories are the following:

• There is a functor from the category of schemes to that of Thomason categories: $X \mapsto \mathbf{Perf}(X)$, where $\mathbf{Perf}(X)$ is the category of perfect complexes of globally finite Tor-amplitude (cf. [TT90], §2.2).

• For an appropriate scheme X, from $\mathbf{Perf}(X)$ (and its tensor structure), we can recover the scheme X completely [Bal02]. That is, $\mathbf{Perf}(X)$ does not lose the geometric information of X.

• Moreover in the category of Thomason categories, we have objects like $\mathbf{Perf}(X \text{ on } Y)$ and $\mathbf{Perf}^r(X)$ (cf. Def. 3.9) which are derived from schemes and absent from the category of schemes.

• Since the algebraic K-theory is ∞-categorical invariant (see [Sch02], [Toe03], [TV04] and [BM07]), we prefer to the category of Thomason categories than that of triangulated categories.

Next we need to consider how to enrich the category V and choose a

relation space W. Inspired from the work [Wal96] and encouraged by the works [Kon07] §4 and [Tau07], the authors intend to enriching V with the bivariant algebraic K-theory. To mention the reason why we like to select the bivariant K-theory as morphisms spaces of V, we will start from the following Lemma 2.12. Let $\mathcal{D}$ be a tensor triangulated category and $\mathrm{M} : \mathbf{qpsmcor}(K) \to \mathcal{D}$ a functor preserving coproducts and tensor products. Here the tensor products in $\mathbf{qpsmcor}(k)$ are the usual products over $\operatorname{Spec} k$. From now on, for $\mathbb{P}^1$-homotopy invariant, we mean that $\mathbb{P}^1$-homotopy invariant for the triple $(\mathbb{P}^1, 0, 1)$.

Lemma 2.12 (Compare [CHK97]). *The following conditions are equivalent.*
(a) M *is* $\mathbb{P}^1$*-homotopy invariant.*
(b) *The following diagram is commutative.*

$$\begin{tikzcd} \mathrm{M}(\mathbb{A}^1 \smallsetminus \{0\}) \arrow[rr, "\mathrm{M}(i)"] \arrow[dr, "\mathrm{M}(p)"'] & & \mathrm{M}(\mathbb{P}^1) \\ & \mathrm{M}(\operatorname{Spec} k) \arrow[ur, "\mathrm{M}(\infty)"'] & \end{tikzcd}$$

where i and p are the natural inclusion and the structure map respectively.
(c) *(Rigidity)* $\mathrm{M}(1) = \mathrm{M}(\infty) : \mathrm{M}(\operatorname{Spec} k) \to \mathrm{M}(\mathbb{P}^1)$ *is coincided.*

Proof. (a) $\Rightarrow$ (b): The map

$$H : (\mathbb{A}^1 \smallsetminus \{0\}) \times \mathbb{P}^1 \ni (t, [x_0 : x_1]) \mapsto [tx_0 : tx_1] \in \mathbb{P}^1$$

gives $\mathbb{P}^1$-homotopy between i and $\infty \circ p$. Therefore we get the results.
(b) $\Rightarrow$ (c): Considering the following diagram, we get the result.

$$\begin{tikzcd} \mathrm{M}(\operatorname{Spec} k) \arrow[r, "\mathrm{M}(0)"] \arrow[d, "\mathrm{id}"] & \mathrm{M}(\mathbb{A}^1 \smallsetminus \{0\}) \arrow[d, "\mathrm{M}(i)"] \arrow[dl, "\mathrm{M}(p)"] \\ \mathrm{M}(\operatorname{Spec} k) \arrow[r, "\mathrm{M}(\infty)"'] & \mathrm{M}(\mathbb{P}^1). \end{tikzcd}$$

(c) $\Rightarrow$ (a): Let f, $g : X \to Y$ be maps in $\mathbf{qpsmcor}(k)$ and $H : X \times \mathbb{P}^1 \to Y$ are their $\mathbb{P}^1$-homotopy, that is, $H \circ 0 \times \mathrm{id}_X = f$ and $H \circ 1 \times \mathrm{id}_X = g$. Then

we have the identity:

$$\begin{aligned}
\mathrm{M}(f) &= \mathrm{M}(H \circ 0 \times \mathrm{id}_X) = \mathrm{M}(H \circ \tau \circ \infty \times \mathrm{id}_X) \\
&= \mathrm{M}(H)\,\mathrm{M}(\tau)\,\mathrm{M}(\infty) \otimes \mathrm{M}(\mathrm{id}_X) = \mathrm{M}(H) M(\tau)\,\mathrm{M}(1) \otimes \mathrm{M}(\mathrm{id}_X) \\
&= \mathrm{M}(H \circ \tau \circ 1 \times \mathrm{id}_X) = \mathrm{M}(H \circ 1 \times \mathrm{id}_X) = \mathrm{M}(g)
\end{aligned}$$

where $\tau : \mathbb{P}^1 \ni [x_0 : x_1] \mapsto [x_1 : x_0] \in \mathbb{P}^1$. □

For the importance of the commutative diagram in Lemma 2.12 (b), the readers shall consult with [CHK97] and this topic will be treated in [HM08]. It is closely related to the existence of the Gersten resolution for M. We also notice that the additive K-theory, additive higher Chow groups and the additive group $\mathbf{G}_a$ are $\mathbb{P}^1$-homotopy invariant as functors on the category of algebraic varieties (see for example [Qui73], [TT90], [Kel99], [KL07]). But K_0 is not a functor on $\mathbf{qpsmcor}(k)$. As in § 2.2, we sometime hope to extend the notion of motivic modules to make functors above belong to the class of generalized motivic modules. Imitating Walker's argument, we prefer to replace $\mathbf{qpsmcor}(k)$ with $K_0^{\mathrm{naive}}(\mathbf{qpsm}(k))$ which is the category of quasi-projective smooth schemes over k enriching with the bivariant K-theory (For precise definition, see [Wal96], [Sus03]). We like to call $\mathbb{P}^1$-homotopy invariant presheaves of abelian groups on $K_0^{\mathrm{naive}}(\mathbf{qpsm}(k))$ *generalized motivic modules*. Now it is important that we recall the following core theorem of the $\mathbb{A}^1$-motivic theory. Let us assume that k is a perfect field.

Theorem 2.13. *For an $\mathbb{A}^1$-homotopy invariant presheaf with transfer F, we have the following.*

(i) *For any p, $\mathrm{H}^p_{\mathrm{Nis}}(?, F_{\mathrm{Nis}})$ can be considered as an $\mathbb{A}^1$-homotopy invariant Nisnevich sheaf with transfer in the natural way. That is, Nisnevich motivic modules are closed under taking cohomology.*

(ii) $\mathrm{H}^p_{\mathrm{Nis}}(?, F_{\mathrm{Nis}}) \xrightarrow{\sim} \mathrm{H}^p_{\mathrm{Zar}}(?, F_{\mathrm{Zar}})$ *for any p.*

Now Beilinson and Vologodsky perceived that Theorem 2.13 is a consequence of the existence of the Gersten resolution of F [BV07] and Walker proved that for $\mathbb{A}^1$-homotopy invariant presheaves on $K_0(\mathbf{qpsm}(k))$, similar theorem above are verified [Wal96]. Therefore the touchstone of a notion of generalized motivic modules are following.

• For a generalized motivic module, does it have the Gersten resolution ?

• Does the (equidimensional) bivariant K-theory have the expected properties like as the Friedlander-Voevodsky theory [FV00] ?

In this paper, the authors prepare to attack to the second problem above. More precisely saying, in this paper and [Moc08], the authors will observe that the roll of a base of our motivic theory, analyze how to avoid to the geometry over the base (see § 2.5). Symbolically, let us denote $\star$ the invisible base for our motivic theory. If there exist a bigraded bivariant K-theory for schemes, in particular we can consider $K_{p,q}(X, \star)$ for a scheme X. The second author believe that $K_{p,q}(X, \star)$ might be $K_p(\mathbf{Wt}^q(X))$ and for a regular noetherian affine scheme X, the isomorphism

$$K_p(\mathbf{Wt}^q(X)) \xrightarrow{\sim} K_p(\mathcal{M}^q(X))$$

could be considered as a variant of Friedlander-Voevodsky duality theorem [FV00].

2.5 How to describe deforming motivic theories II

To compare with two motivic theories, the author intend to parametrize the relation space W in Conjecture 2.11. Namely for example we consider moduli space of motivic theories $V/W(t)$.

Example 2.14. Let R be a commutative discrete valuation ring and π its uniformizer. We put $K = R[1/\pi]$ and $k = R/\pi R$. Then we can consider the *parametrized Susulin functor* by using the following parametrized cosimplicial scheme $\Delta_\bullet$. We define a parametrized cosimplicial scheme $\Delta_\bullet$ by

$$[n] \mapsto \operatorname{Spec} R[T_0, \ldots, T_n]/(\Sigma T_i - \pi).$$

Obviously $\Delta_\bullet|_{\operatorname{Spec} K}$ is usual one and $\Delta_\bullet|_{\operatorname{Spec} k}$ is appeared in [BE03].

The attempt in Example 2.14 is just a naive construction of deformation space of motivic theories parametrized by $\operatorname{Spec} R$ whose fiber over $\operatorname{Spec} K$ is the $\mathbb{A}^1$-motivic theory and over $\operatorname{Spec} k$ is an alien one. But we are confronted with the following serious problems.

What is the motivic theory of total space?

Why does the total space theory work fine?

To solve the second problem above, we need to assure that we can build up a motivic theory without relying upon the geometry over a base. Therefore our deforming motivic theories is starting from examining the Thomason-Trobaugh weight.

3 Preliminary

3.1 Tor-dimension

We briefly review the definition and fundamental properties of *Tor-dimension* of Modules.

Definition 3.1. Let $\mathcal{L}$ be an $\mathcal{O}_X$-Module.
(i) $\mathcal{L}$ is *flat* if the functor $? \otimes_{\mathcal{O}_X} \mathcal{L} : \mathbf{Mod}(X) \to \mathbf{Mod}(X)$ defined by $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$ is exact.
(ii) A *Tor-dimension* of $\mathcal{L}$ is the minimal integer n such that there is a resolution of $\mathcal{L}$,

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_0 \to \mathcal{L} \to 0,$$

where all $\mathcal{F}_i$ are flat. We write as $\mathrm{Td}(\mathcal{L}) = n$.

Now we list some well-known facts on Tor-dimension.

Lemma 3.2 ([BGI71], Exp. I, 5.8.3, [DG63], 6.5.7.1)**.** *Let $\mathcal{L}$ be an $\mathcal{O}_X$-Module.*
(i) *If $\mathcal{L}$ is a flat and finitely presented $\mathcal{O}_X$-Module, then $\mathcal{L}$ is an algebraic vector bundle.*
(ii) *The following conditions are equivalent.*
(a) $\mathrm{Td}(\mathcal{L}) \leqq d$.
(b) *For any $\mathcal{O}_X$-Module $\mathcal{K}$ and any $n > d$, we have* $\mathrm{Tor}_n^{\mathcal{O}_X}(\mathcal{L}, \mathcal{K}) = 0$.
(c) *For any $\mathcal{O}_X$-Module $\mathcal{K}$, we have* $\mathrm{Tor}_{d+1}^{\mathcal{O}_X}(\mathcal{L}, \mathcal{K}) = 0$.
(d) *If there is an exact sequence*

$$0 \to \mathcal{K}_d \to \mathcal{F}_{d-1} \to \mathcal{F}_{d-2} \to \cdots \to \mathcal{F}_0 \to \mathcal{L} \to 0$$

where all $\mathcal{F}_i$ are flat, then $\mathcal{K}_d$ is also flat.
(iii) *For any short exact sequence of $\mathcal{O}_X$-Modules*

$$0 \to \mathcal{L} \to \mathcal{L}' \to \mathcal{L}'' \to 0,$$

we have a formula $\mathrm{Td}(\mathcal{L}') \leqq \max\{\mathrm{Td}(\mathcal{L}),\ \mathrm{Td}(\mathcal{L}'')\}$.

(iv) *For any* $x \in X$ *and quasi-coherent* $\mathcal{O}_X$*-Modules* $\mathcal{L}$, $\mathcal{K}$, *we have*

$$\mathcal{TOR}_n^{\mathcal{O}_X}(\mathcal{L}, \mathcal{K})_x \xrightarrow{\sim} \mathrm{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{K}_x).$$

As its consequence, we have the following formula.

$$\mathrm{Td}(\mathcal{L}) \leqq \sup_{x \in X} \mathrm{Td}_{\mathcal{O}_{X,x}}(\mathcal{L}_x).$$

We define a similar Tor-dimension for unbounded complexes.

Definition 3.3 ([TT90], Def. 2.2.11)**.** Let $E^\bullet$ be a complex of $\mathcal{O}_X$-Modules.

(i) $E^\bullet$ has (*globally*) *finite Tor-amplitude* if there are integers $a \leqq b$ and for all $\mathcal{O}_X$-Module $\mathcal{F}$, $\mathrm{H}^k(E^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{F}) = 0$ unless $a \leqq k \leqq b$. (In the situation, we say that $E^\bullet$ *has Tor-amplitude contained in* $[a, b]$).

(ii) $E^\bullet$ has *locally finite Tor-amplitude* if X is covered by opens U such that $E^\bullet|_U$ has finite Tor-amplitude.

Remark 3.4. (i) If the scheme X is quasi-compact, then every locally finite Tor-amplitude complex $E^\bullet$ of $\mathcal{O}_X$-Modules is globally finite Tor-amplitude.

(ii) For three vertexes of a distinguished triangle in the derived category of $\mathbf{Mod}(X)$, if two of these three vertexes are globally finite Tor-amplitude then the third vertex is also.

3.2 The coherator

We briefly review the theory of "coherator" from [BGI71], II and [TT90] Appendix B. There are two abelian categories $\mathbf{Qcoh}(X)$ and $\mathbf{Mod}(X)$ and the canonical inclusion functor $\phi_X : \mathbf{Qcoh}(X) \hookrightarrow \mathbf{Mod}(X)$ which is exact, closed under extensions, reflects exactness, preserves and reflects infinite direct sums. The problem is that in general ϕ_X does not preserve injective objects in $\mathbf{Qcoh}(X)$. But for coherent schemes, there is a good theory for $\mathbf{Qcoh}(X)$. We are starting from reviewing the definition of coherence of schemes.

Definition 3.5 ([DGSV72], VI)**.** The scheme X is said to be *quasi-separated* if the diagonal map $X \to X \times X$ is quasi-compact or equivalently if intersection of any pair of affine open sets in X is quasi-compact. It is said to be *coherent* if it is quasi-compact and quasi-separated.

Proposition 3.6 ([BGI71], II, 3.2; [TT90], Appendix B)**.** *Let X be a coherent scheme. Then we have the following:*

(i) *ϕ_X has the right adjoint functor $Q_X : \mathbf{Mod}(X) \to \mathbf{Qcoh}(X)$ which is said to be* coherator *and the canonical adjunction map* $\mathrm{id} \to Q_X\phi_X$ *is an isomorphism. In particular $\mathbf{Qcoh}(X)$ has enough injective and closed under limit.*

(ii) *Q_X preserves limit.*

(iii) *For any $E^\bullet \in \mathcal{D}(\mathbf{Qcoh}(X))$ and $F^\bullet \in \mathcal{D}(\mathbf{Mod}(X))$, the canonical adjunction maps $E^\bullet \to RQ_X\phi_X E^\bullet$ and $\phi_X RQ_X F^\bullet \to F^\bullet$ are quasi-isomorphisms.*

3.3 Perfect and pseudo-coherent complexes

We review the notion of pseudo-coherent and perfect complexes. For a complex of $\mathcal{O}_X$-Modules $E^\bullet$ on X, perfection and pseudo-coherence depend only on the quasi-isomorphism class of $E^\bullet$ and are local properties on X. So first we define the strict version of them and next we define them as being local properties.

Definition 3.7 ([BGI71], Exp. I; [TT90], § 2.2)**.** Let $E^\bullet$ be a complex of $\mathcal{O}_X$-Modules.

(i) $E^\bullet$ is *strictly perfect* (resp. *strictly pseudo-coherent*) if it is a bounded complex (resp. bounded above complex) of algebraic vector bundles.

(ii) $E^\bullet$ is *perfect* (resp. *n-pseudo-coherent*) if it is locally quasi-isomorphic (resp. n-quasi-isomorphic) to strictly perfect complexes. More precisely, for any point $x \in X$, there is a neighborhood U in X, a strictly perfect complex $F^\bullet$, and a quasi-isomorphism (resp. an n-quasi-isomorphism) $F^\bullet \xrightarrow{\sim} E^\bullet|_U$. $E^\bullet$ is said to be *pseudo-coherent* if it is n-pseudo-coherent for all integer n.

Lemma 3.8 ([TT90], § 2.2)**.** *Let $E^\bullet$ be a complex of $\mathcal{O}_X$-Modules on X.*

(i) *If $E^\bullet$ is strictly pseudo-coherent, then it is pseudo-coherent.*

(ii) *In general, a pseudo-coherent complex may not be locally quasi-isomorphic to a strictly pseudo-coherent complex. But if $E^\bullet$ is pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-Modules, then $E^\bullet$ is locally quasi-isomorphic to a strictly pseudo-coherent complex.*

(iii) *If $E^\bullet$ is a pseudo-coherent, then all cohomology sheaf $\mathrm{H}^i(E^\bullet)$ is quasi-coherent. In particular, a pseudo-coherent $\mathcal{O}_X$-Module is a quasi-coherent*

$\mathcal{O}_X$*-Module. Moreover if we assume* X *is quasi-compact and* $E^\bullet$ *is pseudo-coherent, then* $E^\bullet$ *is cohomologically bounded above.*

(iv) *Moreover if we assume* X *is noetherian, we have the following equivalent conditions.*

(a) $E^\bullet$ *is pseudo-coherent.*

(b) $E^\bullet$ *is cohomologically bounded above and all the cohomology sheaf* $H^k(E^\bullet)$ *are coherent* $\mathcal{O}_X$*-Modules.*

In particular, a pseudo-coherent $\mathcal{O}_X$*-Module is coherent.*

(v) *The complex* $E^\bullet$ *is perfect if and only if* $E^\bullet$ *is pseudo-coherent and has locally finite Tor-amplitude.*

(vi) *Pseudo-coherence and perfection have* 2 *out of* 3 *properties. Namely, let* $E^\bullet$, $F^\bullet$ *and* $G^\bullet$ *be the three vertexes of a distinguished triangle in the derived category of* $\mathbf{Mod}(X)$ *and if two of these three vertexes are pseudo-coherent (resp. perfect) then the third vertex is also.*

(vii) *For any complexes of* $\mathcal{O}_X$*-Modules* $F^\bullet$ *and* $G^\bullet$, $F^\bullet \oplus G^\bullet$ *is pseudo-coherent (resp. perfect) if and only if* $F^\bullet$ *and* $G^\bullet$ *are.*

(viii) *A strictly bounded complex of perfect* $\mathcal{O}_X$*-Modules* $E^\bullet$ *is perfect.*

Definition 3.9. (i) For any $\mathcal{O}_X$-Module $\mathcal{F}$, we denote its *support* by

$$\operatorname{Supp}\mathcal{F} := \{x \in X; \mathcal{F}_x \neq 0\}.$$

(ii) ([Tho97], 3.2) For a complex of $\mathcal{O}_X$-Modules $E^\bullet$, the *cohomological support* of $E^\bullet$ is the subspace $\operatorname{Supph} E^\bullet \subset X$ those points $x \in X$ at which the stalk complex of $\mathcal{O}_{X,x}$-module $E^\bullet_x$ is not acyclic.

(iii) For any closed subset Y of X, we denote by $\mathbf{Perf}(X \text{ on } Y)$ (resp. $\mathbf{Perf}_{\mathrm{qc}}(X \text{ on } Y)$, $\mathbf{sPerf}(X \text{ on } Y)$) the complicial biWaldhausen category of globally finite Tor-amplitude perfect complexes (resp. globally finite Tor-amplitude perfect complexes of quasi-coherent $\mathcal{O}_X$-Modules, strictly perfect complexes) whose cohomological support on Y. Here, the cofibrations are the degree-wise split monomorphisms, and the weak equivalences are the quasi-isomorphisms. Put

$$\mathbf{Perf}^r(X) := \bigcup_{\operatorname{Codim} Y \geqq r} \mathbf{Perf}(X \text{ on } Y).$$

Lemma 3.10. (i) *For any short exact sequence of* $\mathcal{O}_X$*-Modules*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

we have $\operatorname{Supp}\mathcal{G} = \operatorname{Supp}\mathcal{F} \cup \operatorname{Supp}\mathcal{H}$.

(ii) *For a complex of* $\mathcal{O}_X$*-Modules* $E^\bullet$, *we have* $\operatorname{Supph} E^\bullet = \cup_{n\in\mathbb{Z}} \operatorname{Supp} \mathrm{H}^n(E^\bullet)$.

Lemma 3.11. *For a coherent scheme* X *and its closed set* Y*, the canonical inclusion functor* $\mathbf{Perf}_{\mathrm{qc}}(X \text{ on } Y) \hookrightarrow \mathbf{Perf}(X \text{ on } Y)$ *induces an equivalence of categories between thir derived categories.*

Proof. The inverse functor of $\mathcal{T}(\mathbf{Perf}_{\mathrm{qc}}(X \text{ on } Y)) \to \mathcal{T}(\mathbf{Perf}(X \text{ on } Y))$ is given by the coherator (Prop. 3.6, (iii)). $\square$

3.4 Divisorial schemes

Since perfect and pseudo-coherent complexes are well-behavior on *divisorial* schemes, we briefly review the definition and fundamental properties of divisorial schemes.

Definition 3.12 ([BGI71], II, 2.2.5; [TT90], Def. 2.1.1)**.** A coherent scheme X is said to be *divisorial* if it has *an ample family of line bundles.* That is it has a family of line bundles $\{\mathcal{L}_\alpha\}$ which satisfies the following condition (see *op. cit.*for another equivalent conditions):
For any $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$, we put the open set

$$X_f := \{x \in X | f(x) \neq 0\}.$$

Then $\{X_f\}$ is a basis for the Zariski topology of X where n runs over all positive integer, $\mathcal{L}_\alpha$ runs over the family of line bundles and f runs over all global sections of all of $\mathcal{L}_\alpha^{\otimes n}$.

Example 3.13. (i) A quasi-projective scheme over affine scheme is divisorial. So classical algebraic varieties are divisorial. Since every scheme is locally affine, every scheme is locally divisorial.

(ii) A separated regular noetherian scheme is divisorial.

(iii) ([TT90], Exerc. 8.6) Let k be an field and X an $\mathbb{A}_k^n$ with double origin. Then X is regular noetherian but is not divisorial.

Lemma 3.14 ([DG61], II, 5.5.8 and [BGI71], II, 2.2.3.1)**.** *For a line bundle* $\mathcal{L}$ *on* X *and a section* $f \in \Gamma(X, \mathcal{L})$*, the canonical open immersion* $X_f \to X$ *is affine.*

Theorem 3.15 (Global resolution theorem, [BGI71], II; [TT90], Prop. 2.3.1). *Let X be a divisorial scheme. Then we have the following.*

(i) *Any pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-Modules is globally quasi-isomorphic to a strictly pseudo-coherent complex.*

(ii) *Any perfect complex is isomorphic to a strictly perfect complex in $\mathcal{D}(\mathbf{Mod}(X))$.*

3.5 Regular closed immersion

There are several definitions of regular immersion (see [DG67] and [BGI71], VII). Both definitions are equivalent if a total scheme is noetherian. We adopt the definition in [BGI71] and for readers convenience, we briefly review the notation and fundamental properties of regular closed immersion.

Definition 3.16. Let $u : \mathcal{L} \to \mathcal{O}_X$ be a morphism of $\mathcal{O}_X$-Modules from an algebraic vector bundle $\mathcal{L}$ to $\mathcal{O}_X$. A *Koszul complex* associated to u is the strictly perfect complex $\operatorname{Kos}^\bullet(u)$ defined as follows: For $n > 0$, we put

$$\operatorname{Kos}^{-n}(u)(= \operatorname{Kos}_n(u)) := \bigwedge^n \mathcal{L}, \quad \text{and}$$
$$d_n(x_1 \wedge \cdots \wedge x_n) := \sum_{r=1}^{n} (-1)^{r-1} u(x_r) x_1 \wedge \cdots \wedge \widehat{x}_r \wedge \cdots \wedge x_n.$$

Definition 3.17 ([BGI71], VII, 1.4). (i) An $\mathcal{O}_X$-Module homomorphism $u : \mathcal{L} \to \mathcal{O}_X$ from an algebraic vector bundle $\mathcal{L}$ to $\mathcal{O}_X$ is said to be *regular* if $\operatorname{Kos}^\bullet(u)$ is a resolution of $\mathcal{O}_X / \operatorname{Im} u$.

(ii) An ideal sheaf $\mathcal{I}$ on X is *regular* if locally on X, there is a regular map $u : \mathcal{L} \to \mathcal{O}_X$ such that $\operatorname{Im} u = \mathcal{I}$. More precisely, this means that if there is an open covering $\{U_i\}_{i \in I}$ of X and for each $i \in I$, there is a regular map $u_i : \mathcal{L}\,|_{U_i} \to \mathcal{O}_{U_i}$ such that $\operatorname{Im} u_i = \mathcal{I}\,|_{U_i}$.

(iii) A closed immersion $Y \hookrightarrow X$ is said to be *regular* if the defining ideal of Y is regular. We put $\mathcal{N}_{X/Y} := \mathcal{I} / \mathcal{I}^2$ and call it the *conormal sheaf* of the regular closed immersion.

Lemma 3.18 ([DG67]). *Let $Y \hookrightarrow X$ be a regular closed immersion whose defining ideal is $\mathcal{I}$.*

(i) *The ideal sheaf $\mathcal{I}$ satisfies the following conditions:*

(a) $\mathcal{I}$ *is of finite type.*
(b) *For each* n, $\mathcal{I}^n \,/\, \mathcal{I}^{n+1}$ *is a locally free* $\mathcal{O}_X \,/\, \mathcal{I}$*-Module of finite type.*
(c) *A canonical map*

$$\mathrm{Sym}_{\mathcal{O}_X \,/\, \mathcal{I}}(\mathcal{N}_{X/Y}) \to \mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X)$$

is an isomorphism of $\mathcal{O}_X \,/\, \mathcal{I}$*-Algebra. Here* $\mathrm{Sym}_{\mathcal{O}_X \,/\, \mathcal{I}}(\mathcal{N}_{X/Y})$ *is the symmetric algebra associated to* $\mathcal{N}_{X/Y}$, $\mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n \geqq 0} \mathcal{I}^n \,/\, \mathcal{I}^{n+1}$ *is the graded algebra associated to an* $\mathcal{I}$*-adic filtration in* $\mathcal{O}_X$ *and the canonical map is defined by the universal property of symmetric algebra.*

(ii) *If the scheme* X *is noetherian, then* $\mathcal{I}$ *is regular in the sense of [DG67]. That is, for any point* $x \in X$ *there is an open neighborhood* U *of* x*, and a regular sequence* $f_1, \dots, f_r \in \Gamma(U, \mathcal{I})$ *which generates* $\mathcal{I}\,|_U$.

4 Weight on pseudo-coherent Modules

Definition 4.1. A pseudo-coherent $\mathcal{O}_X$-Module $\mathcal{F}$ is *of weight* r if it is of Tor-dimension $\leqq r$ and there is a regular closed immersion $Y \hookrightarrow X$ of codimension r in X such that the support of $\mathcal{F}$ is in Y.

We denote by $\mathbf{Wt}^r(X)$ the category of pseudo-coherent $\mathcal{O}_X$-Modules of weight r. For a regular closed immersion $Y \hookrightarrow X$ of codimension r, we denote by $\mathbf{Wt}^r(X \text{ on } Y)$ the category of pseudo coherent $\mathcal{O}_X$-Modules of weight r supported on the subspace Y. Immediately, a pseudo-coherent $\mathcal{O}_X$-Module of weight 0 is just an algebraic vector bundle.

Lemma 4.2. *The category* $\mathbf{Wt}^r(X \text{ on } Y)$ *is closed under extensions and direct summand in the abelian category* $\mathbf{Mod}(X)$*. In particular,* $\mathbf{Wt}^r(X \text{ on } Y)$ *is an idempotent complete exact category.*

Proof. The assertion follows from Lemma 3.2, (iii), Lemma 3.10, (i), and Lemma 3.8, (vii). □

A pseudo-coherent $\mathcal{O}_X$-Module $\mathcal{F}$ of weight r has globally finite Tor-amplitude. Thus it is perfect by Lemma 3.8, (v) and we have an inclusion functor $\mathbf{Wt}^r(X \text{ on } Y) \hookrightarrow \mathbf{Perf}(X \text{ on } Y)$. Moreover we have the natural inclusion functor $\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)) \hookrightarrow \mathbf{Perf}(X \text{ on } Y)$ by Lemma 3.8, (viii). Now, we state our main theorem.

Theorem 4.3. *Let X be a divisorial scheme and $Y \hookrightarrow X$ a regular closed immersion of codimension r. Then the inclusion $\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)) \hookrightarrow \mathbf{Perf}(X \text{ on } Y)$ induces a derived Morita equivalence.*

Now consider the inclusion functor $\mathbf{Wt}^r(X \text{ on } Y) \to \mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y))$ which sends $\mathcal{F}$ in $\mathbf{Wt}^r(X \text{ on } Y)$ to the complex which is $\mathcal{F}$ in degree 0 and 0 in other degrees. We denote by $K^S(\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)); \mathrm{qis})$ the K-theory spectrum of the Waldhausen category associated to $\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y))$ whose weak equivalences are the quasi-isomorphisms. The inclusion above induces a homotopy equivalence

$$K^S(\mathbf{Wt}^r(X \text{ on } Y)) \xrightarrow{\sim} K^S(\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)); \mathrm{qis})$$

by non-connected version of the Gillet-Waldhausen theorem in [Sch04]. Therefore we get the following corollary.

Corollary 4.4. *In the notation above, we have the identities*

$$K^S(\mathbf{Wt}^r(X \text{ on } Y)) \xrightarrow{\sim} K^S(X \text{ on } Y) \xrightarrow{\sim} K^B(X \text{ on } Y),$$
$$HC(\mathbf{Wt}^r(X \text{ on } Y)) \xrightarrow{\sim} HC(X \text{ on } Y).$$

Proof. For the K-theory case, it is followed from the observation above and the Schlichting approximation theorem and the comparison theorem in [Sch06]. For the cyclic homology case, it is followed from the derived invariance by [Kel99]. $\square$

5 Proof of the main theorem

First we consider the following two categories. Let $\mathcal{B}$ be the category of perfect complexes in $\mathbf{Ch}^-(\mathbf{Wt}^r(X \text{ on } Y))$ and $\mathcal{C}$ the category of perfect complexes of quasi-coherent $\mathcal{O}_X$-Modules supported on Y. By Lemma 3.8, the categories $\mathcal{B}$ and $\mathcal{C}$ are closed under extensions and direct summand in $\mathbf{Ch}(\mathbf{Mod}(X))$. Therefore, they are idempotent complete exact categories. Note that any perfect complex has globally finite Tor-amplitude on X (Rem. 3.4 and Lem. 3.8, (v)). From Lemma 3.8, (iii), we have the following natural exact inclusion functors

$$\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)) \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \xrightarrow{\gamma} \mathbf{Perf}(X \text{ on } Y).$$

We shall prove α, β and γ induce category equivalences between their associated derived categories by using the following criterion.

Lemma 5.1 ([TT90], 1.9.7 and [Tho93])**.** *Let $i : \mathcal{X} \to \mathcal{Y}$ be a fully faithful complicial exact functor between complicial biWaldhausen categories which closed under the formation of canonical homotopy pullbacks and pushouts and assume their weak equivalence classes are just quasi-isomorphism classes. If i satisfies the condition* (**DE**) *or* (**DE**)$^{\mathrm{op}}$ *below, then i induces category equivalences between their derived categories.*
(**DE**) *For any object Y in $\mathcal{Y}$, there is an object X in $\mathcal{X}$ and a weak equivalence $i(X) \to Y$.*
(**DE**)$^{\mathrm{op}}$ *For any object Y in $\mathcal{Y}$, there is an object X in $\mathcal{X}$ and a weak equivalence $Y \to i(X)$.*

We shall prove that α induces category equivalence between their derived categories. To do so first we review the following lemma.

Lemma 5.2 ([BS01], 2.6)**.** *Let $\mathcal{E}$ be an idempotent complete exact categories and $f : X^\bullet \to Y^\bullet$ a quasi-isomorphism between bounded above complexes in $\mathbf{Ch}(\mathcal{E})$. Assume $X^\bullet$ or $Y^\bullet$ is strictly bounded. Say the other one as $Z^\bullet$. Then there is a sufficiently small N such that $Z^\bullet \to \tau^{\geqq N} Z^\bullet$ is a quasi-isomorphism.*

Lemma 5.3. *The inclusion $\alpha : \mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)) \hookrightarrow \mathcal{B}$ satisfies the condition* (**DE**)$^{\mathrm{op}}$ *in Lemma 5.1. In particular, we have an equivalence of categories*

$$\mathcal{T}(\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y))) \xrightarrow{\sim} \mathcal{T}(\mathcal{B}).$$

Proof. Let $\mathcal{E}$ be the category of pseudo-coherent $\mathcal{O}_X$-Modules of Tor-dimension $\leqq r$. It is closed under extensions (Lem. 3.2, (iii)) and direct summand (Lem. 3.8, (vii)). In particular, it is an idempotent complete exact category. We denote by $\mathcal{D}$ the category of perfect complexes in $\mathbf{Ch}^-(\mathcal{E})$ whose cohomological support is in Y. Fix a complex $P^\bullet$ in $\mathcal{B}$. By the global resolution theorem (Th. 3.15), $P^\bullet$ is quasi-isomorphic to a strict perfect complex. Since we have an inclusion $\mathbf{sPerf}(X \text{ on } Y) \subset \mathcal{D}$, $P^\bullet$ is quasi-isomorphic to a bounded complex in $\mathcal{D}$. Now applying Lemma 5.2 to $\mathcal{E}$, there exists an integer N such that the canonical map $P^\bullet \to \tau^{\geqq N} P^\bullet$ is a quasi-isomorphism. Since $\mathrm{Supp}(\mathrm{Im}\, d^{N-1})$ is in Y, $\tau^{\geqq N} P^\bullet$ is actually in $\mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y))$. The assertion follows from it. □

Proposition 5.4. *The inclusion functor $\beta : \mathcal{B} \hookrightarrow \mathcal{C}$ satisfies the condition* (**DE**) *in Lemma 5.1.*

To prove Proposition 5.4, we need the following lemmas.

Lemma 5.5. (i) *Let $\mathcal{I}$ be the definition ideal of Y. Then $\mathcal{O}_X/\mathcal{I}^p$ is of weight r for any non-negative integer p.*
(ii) *Let $\mathcal{F}$ be a pseudo-coherent $\mathcal{O}_X$-Module of weight r and $\mathcal{L}$ an algebraic vector bundle. Then, $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$ is of weight r.*

Proof. (i) First we notice that $\mathcal{O}_X \,/\, \mathcal{I}$ is in $\mathbf{Wt}^r(X \text{ on } Y)$ by Koszul resolution. Next since $\mathcal{I}^n \,/\, \mathcal{I}^{n+1}$ is locally isomorphic to direct sum of $\mathcal{O}_X \,/\, \mathcal{I}$, we learn that $\mathcal{I}^n \,/\, \mathcal{I}^{n+1}$ is also in $\mathbf{Wt}^r(X \text{ on } Y)$ by Lemma 3.2 (iv). Using Lemma 4.2 for

$$0 \to \mathcal{I}^{n+1} \,/\, \mathcal{I}^{n+p} \to \mathcal{I}^n \,/\, \mathcal{I}^{n+p} \to \mathcal{I}^n \,/\, \mathcal{I}^{n+1} \to 0,$$

the dévissage argument shows that $\mathcal{I}^n \,/\, \mathcal{I}^{n+p}$ is also in $\mathbf{Wt}^r(X \text{ on } Y)$ for any non-negative integer n and positive integer p.
(ii) Since $\mathcal{L}$ is flat, we have an inequality $\operatorname{Td}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}) \leqq r$. We also have a formula

$$\operatorname{Supp} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F} = \operatorname{Supp} \mathcal{L} \cap \operatorname{Supp} \mathcal{F} \subset Y.$$

Therefore $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$ is of weight r. □

Lemma 5.6 ([TT90], Lem. 1.9.5)**.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a full sub additive category of $\mathcal{A}$. Let $\mathcal{C}$ be a full subcategory of $\mathbf{Ch}(\mathcal{A})$ satisfies the following conditions:*
(a) *$\mathcal{C}$ is closed under quasi-isomorphisms. That is, any complex quasi-isomorphic to an object in $\mathcal{C}$ is also in $\mathcal{C}$.*
(b) *Every complex in $\mathcal{C}$ is cohomologically bounded above.*
(c) *$\mathbf{Ch}^b(\mathcal{D})$ is contained in $\mathcal{C}$.*
(d) *$\mathcal{C}$ contains the mapping cone of any map from an object in $\mathbf{Ch}^b(\mathcal{D})$ to an object in $\mathcal{C}$.*
Finally, Suppose the following condition, so "$\mathcal{D}$ has enough objects to resolve":
(e) *For any integer n, any $C^\bullet$ in $\mathcal{C}$ such that $\mathrm{H}^i(C^\bullet) = 0$ for any $i \geqq n$ and any epimorphism in $\mathcal{A}$, $A \twoheadrightarrow \mathrm{H}^{n-1}(C^\bullet)$, then there exists a D in $\mathcal{D}$ and a morphism $D \to A$ such that the composite $D \twoheadrightarrow \mathrm{H}^{n-1}(C^\bullet)$ is an epimorphism in $\mathcal{A}$.*

Then, for any $D^\bullet$ in $\mathbf{Ch}^-(\mathcal{D}) \cap \mathcal{C}$, any $C^\bullet$ in $\mathcal{C}$, and any morphism $x : D^\bullet \to C^\bullet$, there exist a $E^\bullet$ in $\mathbf{Ch}^-(\mathcal{D}) \cap \mathcal{C}$, a degree-wise split monomorphism $a : D^\bullet \to E^\bullet$ and a quasi-isomorphism $y : E^\bullet \xrightarrow{\sim} C^\bullet$ such that $x = y \circ a$. Moreover if $x : D^\bullet \to C^\bullet$ is an n-quasi-isomorphism for some integer n, then one may choose $E^\bullet$ above so that $a^k : D^k \to E^k$ is an isomorphism for $k \geqq n$.

Lemma 5.7. *Let X be a divisorial scheme whose ample family of line bundles is $\{\mathcal{L}_\alpha\}$ and $E^\bullet$ a perfect complex on X. Then there are line bundles $\mathcal{L}_{\alpha_k}$ in the ample family, integers m_k and sections $f_k \in \Gamma(X, \mathcal{L}_{\alpha_k}^{\otimes m_k})$ $(1 \leqq k \leqq m)$ such that*
(a) *For each k, X_{f_k} is affine.*
(b) *$\{X_{f_k}\}_{1 \leqq k \leqq m}$ is an open cover of X.*
(c) *For each k, $E^\bullet|_{X_{f_k}}$ is quasi-isomorphic to a strictly perfect complex.*

Proof. Since $E^\bullet$ is perfect, we can take an affine open covering $\{U_i\}_{i\in I}$ of X such that $E^\bullet|_{U_i}$ is quasi-isomorphic to a strictly perfect complex for each $i \in I$. Since $\{\mathcal{L}_\alpha\}$ is an ample family, for each $x \in$X, there are an $i_x \in I$, a line bundle $\mathcal{L}_{\alpha_x}$ in the ample family, an integer m_x and a section $f_x \in \Gamma(X, \mathcal{L}_{\alpha_x}^{\otimes m_x})$ such that $x \in X_{f_x} \subset U_{i_x}$. Since U_{i_x} is affine, X_{f_x} is affine by Lemma 3.14. Now $\{X_{f_x}\}_{x \in X}$ is an affine open covering of X and has a finite sub covering by quasi-compactness of X. $\square$

Lemma 5.8 ([TT90], Lem. 1.9.4, (b))**.** *Let $E^\bullet$ be a strictly pseudo-coherent complex on X such that $\mathrm{H}^i(E^\bullet) = 0$ for $i \geqq m$. Then $\operatorname{Ker} d^{m-1}$ is an algebraic vector bundle. In particular $\mathrm{H}^{m-1}(E^\bullet)$ is of finite type.*

Proof of Prop. 5.4. Let $\{\mathcal{L}_\alpha\}$ be an ample family of line bundles on X and $\mathcal{I}$ the defining ideal of Y. We denote by $\mathcal{D}$ the additive category generated by all the $\mathcal{L}_\alpha^{\otimes m} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}^p$ with integer m and positive integer p. By Lemma 5.5, $\mathcal{D} \subset \mathbf{Wt}^r(X \text{ on } Y)$. We intend to apply Lemma 5.6 to $\mathcal{A} = \mathbf{Qcoh}(X \text{ on } Y)$ the category of quasi-coherent $\mathcal{O}_X$-Modules whose support on Y. To do so, we have to check the assumptions in Lemma 5.6. Only non-trivial assumption is "having enough objects to resolve" condition. Let $C^\bullet$ be a complex in $\mathcal{C}$ such that $\mathrm{H}^i(C^\bullet) = 0$ for $i \geqq n$, and $\mathcal{F} \twoheadrightarrow \mathrm{H}^{n-1}(C^\bullet)$ an epimorphism in $\mathcal{A}$. By Lemma 5.7, there are line bundles $\mathcal{L}_{\alpha_k}$ integers m_k and their sections $f_k \in \Gamma(X, \mathcal{L}_{\alpha_k}^{\otimes m_k})$ $(1 \leqq k \leqq m)$ such that they satisfy the following conditions.
(a) For each k, X_{f_k} is affine.
(b) $\{X_{f_k}\}_{1 \leqq k \leqq m}$ is an open cover of X.
(c) For each k, $C^\bullet|_{X_{f_k}}$ is quasi-isomorphic to a strictly perfect complex.
Fix an integer k. Since $\mathrm{H}^{n-1}(C^\bullet)|_{X_{f_k}}$ is of finite type by Lemma 5.8, there is sub $\mathcal{O}_{X_{f_k}}$-Module of finite type $\mathcal{G} \subset \mathcal{F}|_{X_{f_k}}$ such that the composition $\mathcal{G} \hookrightarrow \mathcal{F}|_{X_{f_k}} \twoheadrightarrow \mathrm{H}^{n-1}(C^\bullet)|_{X_{f_k}}$ is an epimorphism. Now since $\mathcal{G}$ and $\mathcal{I}|_{X_{f_k}}$ are $\mathcal{O}_{X_{f_k}}$-Modules of finite type (Lemma 3.18, (i)), we have $(\mathcal{I}|_{X_{f_k}})^{p_k}\mathcal{G} = 0$ for some p_k. Therefore $\mathcal{G}$ is considered as $\mathcal{O}_X/\mathcal{I}^{p_k}|_{X_{f_k}}$-Module of finite type.

Hence we have an epimorphism $(\mathcal{O}_X/\mathcal{I}^{p_k}|_{X_{f_k}})^{\oplus t_k} \twoheadrightarrow \mathcal{G}$. We have an $\mathcal{O}_X$-Modules homomorphism $(\mathcal{O}_X/\mathcal{I}^{p_k})^{\oplus t_k} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_{\alpha_k}^{\otimes m_k s_k}$ for some integer s_k ([DG60], 9.3.1 and [DG64], 1.7.5). Therefore considering the same argument for every k, we get a morphism

$$\bigoplus_{k=1}^{m} (\mathcal{O}_X/\mathcal{I}^{p_k} \otimes_{\mathcal{O}_X} \mathcal{L}_{\alpha_k}^{\otimes -m_k s_k})^{\oplus t_k} \to \mathcal{F}$$

whose composition with $\mathcal{F} \twoheadrightarrow \mathrm{H}^{n-1}(C^{\bullet})$ is an epimorphism in $\mathbf{Qcoh}(X \text{ on } Y)$. □

Finally, we shall prove that γ induces category equivalence between their derived categories. Now we consider the following exact inclusion functors:

$$\mathcal{C} \xrightarrow{\gamma_1} \mathbf{Perf}_{\mathrm{qc}}(X \text{ on } Y) \xrightarrow{\gamma_2} \mathbf{Perf}(X \text{ on } Y).$$

Lemma 3.11 assert that γ_2 induces a homotopy equivalence on spectra. Thus, it is enough to show that the inclusion functor γ_1 induces an equivalence of categories between their derived categories. More strongly we show the following:

Lemma 5.9. *The local cohomological functor*

$$R\Gamma_Y = \varinjlim \mathcal{EXT}(\mathcal{O}_X/\mathcal{I}^p, ?) : \mathcal{T}(\mathbf{Perf}_{\mathrm{qc}}(X \text{ on } Y)) \to \mathcal{T}(\mathcal{C})$$

gives inverse functor of the inclusion functor γ_1.

Proof. Let us consider the functor

$$\Gamma_Y := \varinjlim \mathcal{HOM}(\mathcal{O}_X/\mathcal{I}^p, ?) : \mathbf{Qcoh}(X) \to \mathbf{Qcoh}(X \text{ on } Y).$$

Since $\mathcal{I}$ is of finite type, for any $\mathcal{O}_X$-Module $\mathcal{M}$ in $\mathbf{Qcoh}(X \text{ on } Y)$, we have the identity

$$\Gamma_Y \mathcal{M} = \mathcal{M}\,. \tag{3}$$

This identity and the existence of the canonical natural transformation $\Gamma_Y \to \mathrm{id}$ imply that Γ_Y is a right adjoint functor of the inclusion $\mathbf{Qcoh}(X \text{ on } Y) \hookrightarrow \mathbf{Qcoh}(X)$. Therefore we learn that $\mathbf{Qcoh}(X \text{ on } Y)$ has enough injective objects and for any complex $E^{\bullet}$ in $\mathcal{C}$ such that each components are injective quasi-coherent $\mathcal{O}_X$-Modules, we have the identity $R\Gamma_Y E^{\bullet} = E^{\bullet}$ by (3). Combining the obvious fact that γ_1 is fully faithful, we conclude that $R\Gamma_Y$ gives an inverse functor of γ_1. □

6 Applications

In this section, we assume that A is the Cohen-Macaulay ring of Krull dimension d and $X = \operatorname{Spec} A$. By the very definition, the ring A satisfies the following condition (cf. [Bou98], §2.5, Prop. 7): For any ideal J in A such that its hight $\operatorname{ht} J = r$, there is an A-regular sequence $x_1, \ldots, x_r$ contained in J.

In this case, a coherent A-module of weight d is just a module of finite length and finite projective dimension.

Proposition 6.1. *For any integer $0 \leqq r \leqq d$, $\mathbf{Wt}^r(X)$ is closed under extensions in $\mathbf{Mod}(X)$. In particular $\mathbf{Wt}^r(X)$ is an idempotent complete exact category in the natural way.*

Proof. Let us consider the short exact sequence

$$\mathcal{F} \rightarrowtail \mathcal{G} \twoheadrightarrow \mathcal{H}$$

in $\mathbf{Mod}(X)$ such that $\mathcal{F}$ and $\mathcal{H}$ are in $\mathbf{Wt}^r(X)$. Then we learn that $\mathcal{G}$ is of Tor-dimension $\leqq r$ and $\operatorname{Codim} \operatorname{Supp} \mathcal{G} \geqq r$. Therefore there is an A-regular sequence $x_1, \ldots, x_r$ such that $\operatorname{Supp} \mathcal{G} \subset V(x_1, \ldots, x_r)$. Hence we conclude that $\mathcal{G}$ is in $\mathbf{Wt}^r(X)$. $\square$

Theorem 6.2. *For any integer $0 \leqq r \leqq d$, the canonical inclusion functor $\mathbf{Ch}^b(\mathbf{Wt}^r(X)) \hookrightarrow \mathbf{Perf}^r(X)$ is a derived Morita equivalence.*

Proof. We can write the categories $\mathbf{Ch}^b(\mathbf{Wt}^r(X))$ and $\mathbf{Perf}^r(X)$ as follows.

$$\mathbf{Ch}^b(\mathbf{Wt}^r(X)) = \lim_{Y \subset X} \mathbf{Ch}^b(\mathbf{Wt}^r(X \text{ on } Y)),$$
$$\mathbf{Perf}^r(X) = \lim_{Y \subset X} \mathbf{Perf}(X \text{ on } Y),$$

where the limits taking over the regular closed immersion of codimension $\geqq r$. Hence we get the result by Theorem 4.3 and continuity of functor $\mathcal{T}$. $\square$

Corollary 6.3. *For any integer $0 \leqq r \leqq d$, we have the canonical homotopy equivalence of spectra and mixed complexes*

$$K^S(\mathbf{Wt}^r(X)) \xrightarrow{\sim} K^S(\mathbf{Perf}^r(X)),$$
$$HC(\mathbf{Wt}^r(X)) \xrightarrow{\sim} HC(\mathbf{Perf}^r(X)).$$

Proof. Since both theories are derived invariant, the statement is just a corollary of Theorem 6.2. □

Now moreover we assume that A is local and let $\mathfrak{m}$ be its maximal ideal. Then since A is Cohen-Macaulay, $Y := V(\mathfrak{m}) \hookrightarrow X$ is a regular closed immersion. Therefore by Theorem 4.3, we learn that $K^S(X \text{ on } Y)$ is homotopy equivalent to $K^S(\mathbf{Wt}(X \text{ on } Y))$. Now recall that Weibel's K-dimensional conjecture.

Conjecture 6.4 (K-dimensional conjecture)**.** *For any noetherian scheme Z of finite Krull-dimension n, and integer $q > n$, we have $K^B_{-q}(Z) = 0$.*

This conjecture is recently proved for schemes which is essentially of finite type over a field of characteristic 0 [CHSW08]. According to the paper [Bal07], if for any local ring $\mathcal{O}_{Z,z}$ of Z, we have $K^B_{-q}(\operatorname{Spec}\mathcal{O}_{Z,z} \text{ on } \overline{\{z\}}) = 0$ for $q > \dim\mathcal{O}_{Z,z}$, then the conjecture above is true for Z. Therefore for any Cohen-Macaulay scheme, the conjecture is reduced to vanishing of $K^S_{-q}(\mathbf{Wt}(X \text{ on } Y))$ for $q > d$.

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